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ON THE THEORY OF ANISOTROPIC ELASTIC SHELLS AND PLATES

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Abstract—This paper presents a linear theory of homogeneous anisotropic elastic plates and shells, established without considering the Love–Kirchhoff assumptions.†

The boundary conditions on the external bounding surfaces of the shell are rigorously satisfied. No restriction is made as regards the thickness of the shell which permits a study of thick plates and shells. Finally, with the aid of the results obtained in the first part, the problem of anisotropic elastic plates is examined.

1. GEOMETRICAL CONSIDERATIONS

THE shell may be defined as a region of space, bounded by two surfaces $s^{\pm}(z = \pm h/2)$, symmetrically placed with respect to the middle surface s, $(z = 0)$, and to a lateral cylindrical surface Σ with the generators parallel to the z axis. The position vector of a point of the shell space may be defined by

$$\mathbf{R}(x^1, x^2, x^3) = \mathbf{r}(x^1, x^2) + x^3\mathbf{n} \tag{1.1}$$

where $\mathbf{n}$ is the unit vector, $x^3 = z(-h/2 \leqslant z \leqslant h/2)$ the distance of the respective point to the middle surface, h the constant thickness of the shell, $\mathbf{r} = \mathbf{r}(x^\alpha)$ the position vector of an arbitrary point of the middle surface, and x^μ $(\mu = 1, 2)$‡ the Gaussian coordinates of the points of surface. We denote by

$$\mathbf{a}_\rho = \frac{\partial \mathbf{r}}{\partial x^\rho}, \qquad a_{\alpha\beta} = \mathbf{a}_\alpha \mathbf{a}_\beta \quad \text{and} \quad \mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial x^i}, \qquad g_{ij} = \mathbf{g}_i \mathbf{g}_j$$

the vectors of the covariant basis and the covariant components, of the metric tensor corresponding to the middle surface and to the shell space respectively.

In the following, the partial derivative of a vector or tensor with respect to the coordinate x^j will be indicated by the subscript j preceded by a comma, the space covariant derivative will be indicated by a double vertical line and the surface covariant derivative by a Greek subscript preceded by a single vertical line.

Taking into account that the space Christoffel-symbols of the second kind Γ_{ij}^k may be expressed in terms of corresponding surface quantities by relations [3, 4],

† A comprehensive discussion of the contents and error involved in the Love–Kirchhoff assumptions has been given by Koiter [1, 2].

‡ Throughout this paper Greek indices take the values 1, 2 and Latin indices the values 1, 2, 3.

$$\begin{aligned}
\Gamma^{\omega}_{\alpha\beta} &= \overset{0}{\Gamma}{}^{\omega}_{\alpha\beta} - zb^{\omega}_{\beta|\alpha} - z^2 b^{\omega}_{\delta} b^{\delta}_{\alpha|\beta} + \ldots + z^n[\overset{(n)}{g}{}^{\omega\pi}\overset{0}{\Gamma}_{\alpha\beta,\pi} - \overset{(n-1)}{g}{}^{\omega\pi}(b_{\beta\pi|\alpha} + 2\overset{0}{\Gamma}{}^{\tau}_{\alpha\beta} b_{\tau\pi}) \\
&\quad + \overset{(n-2)}{g}{}^{\omega\pi}(b_{\rho\pi} b^{\rho}_{\alpha|\beta} + \overset{0}{\Gamma}{}^{\rho}_{\alpha\beta} c_{\rho\pi})] + \ldots, \\
\Gamma^{3}_{\alpha\beta} &= b_{\alpha\beta} - zc_{\alpha\beta}, \\
\Gamma^{\omega}_{\alpha 3} &= b^{\omega}_{\alpha} - zc^{\omega}_{\alpha} - z^2 b^{\lambda}_{\alpha} c^{\omega}_{\lambda} - \ldots - z^n(\overset{(n)}{g}{}^{\omega\lambda} b_{\alpha\lambda} - \overset{(n-1)}{g}{}^{\omega\lambda} c_{\alpha\lambda}) - \ldots, \\
\Gamma^{\alpha}_{33} &= \Gamma^{3}_{33} = \Gamma^{3}_{\alpha 3} = 0.
\end{aligned} \tag{1.2}\dagger$$

where $\overset{0}{\Gamma}{}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}(x^{\alpha}, 0)$ are the surface Christoffel-symbols of the second kind, $b_{\alpha\beta}$ and $c_{\alpha\beta} = b^{\lambda}_{\alpha} b_{\lambda\beta}$ the second and the third fundamental tensor of the middle surface respectively,

$$\overset{(n)}{g}{}^{\omega\pi} = (n+1)a^{\omega\rho}(b^n)^{\pi}_{\rho} \tag{1.3}$$

are the coefficients of z^n of the expansion into series of $g^{\omega\pi}$ where [5]

$$\begin{aligned}
(b^0)^{\alpha}_{\beta} &= \delta^{\alpha}_{\beta}, \qquad (b^1)^{\alpha}_{\beta} = b^{\alpha}_{\beta}, \\
(b^n)^{\alpha}_{\beta} &= b^{\lambda}_{\beta}(b^{(n-1)})^{\alpha}_{\lambda} = b^{\alpha}_{\lambda}(b^{(n-1)})^{\lambda}_{\beta}
\end{aligned} \tag{1.4}$$

(δ^i_j being the Kronecker symbol), the spatial derivatives of a spatial vector may be expressed with the aid of the surface derivatives as follows [3, 4]

$$\begin{aligned}
V_{\alpha\|\beta} &= V_{\alpha|\beta} - V_3 b_{\alpha\beta} + z(V_3 c_{\alpha\beta} + V_{\omega} b^{\omega}_{\alpha|\beta}) + z^2 V_{\omega} b^{\omega}_{\delta} b^{\delta}_{\alpha|\beta} + \ldots \\
&\quad + \ldots - z^n V_{\omega}[\overset{(n)}{g}{}^{\omega\pi}\overset{0}{\Gamma}_{\alpha\beta,\pi} - \overset{(n-1)}{g}{}^{\omega\pi}(b_{\beta\pi|\alpha} + 2\overset{0}{\Gamma}{}^{\tau}_{\alpha\beta} b_{\tau\pi}) \\
&\quad + \overset{(n-2)}{g}{}^{\omega\pi}(b_{\rho\pi} b^{\rho}_{\alpha|\beta} + \overset{0}{\Gamma}{}^{\rho}_{\alpha\beta} c_{\rho\pi})] + \ldots, \\
V_{\alpha\|3} &= V_{\alpha,3} + V_{\omega}[b^{\omega}_{\alpha} + zc^{\omega}_{\alpha} + z^2 b^{\gamma}_{\alpha} c^{\omega}_{\gamma} + \ldots + z^n(\overset{(n)}{g}{}^{\omega\lambda} b_{\alpha\lambda} - \overset{(n-1)}{g}{}^{\omega\lambda} c_{\alpha\lambda}) + \ldots], \\
V_{3\|\alpha} &= V_{3|\alpha} + V_{\omega}[b^{\omega}_{\alpha} + zc^{\omega}_{\alpha} + z^2 b^{\gamma}_{\alpha} c^{\omega}_{\gamma} + \ldots + z^n(\overset{(n)}{g}{}^{\omega\lambda} b_{\alpha\lambda} - \overset{(n-1)}{g}{}^{\omega\lambda} c_{\alpha\lambda}) + \ldots].
\end{aligned} \tag{1.5}$$

2. PHYSICAL EQUATIONS

The stress–strain equations or vice-versa, written for an elastic homogeneous and anisotropic body are given by

$$\tau^{ij} = E^{ijkl} e_{kl}, \qquad (e_{kl} = F_{klmn}\tau^{mn}) \tag{2.1}$$

where τ^{ij} is the symmetrical stress tensor, e_{ij} the strain tensor, and E^{ijmn}, F_{ijkl} the tensors of elasticity moduli of the body, whose symmetry and homogeneity properties are given in [6]. The conditions which define the anisotropy of the type of the elastic symmetry with respect the surface x^3 = const., or the orthotropy are given in [6].

† All the developments into series of positive integer powers of the variable $x^3 = z$ are assumed to be absolutely and uniformly convergent within the interval $[-h/2, h/2]$.

The magnitudes affected by a negative order number $(n-i)$ will be assumed equal to zero $(\overset{(-k)}{P}{}^{\alpha\beta} = 0$, if $k > 0)$.

In the case of a transversely isotropic body, we have

$$F_{\alpha\beta\omega\rho} = \frac{1+\nu}{E}\left[\frac{1}{2}\left(g_{\alpha\omega}g_{\beta\rho}+g_{\alpha\rho}g_{\beta\omega}\right)-\frac{\nu}{1+\nu}g_{\alpha\beta}g_{\omega\rho}\right];$$

$$F_{\alpha\beta 33} = -\frac{\nu'}{E'}g_{\alpha\beta}; \qquad F_{\alpha 3\omega 3} = \frac{1}{4G'}g_{\alpha\omega}; \qquad F_{3333} = \frac{1}{E'}$$

and

$$E^{\omega\rho\alpha\beta} = \frac{E}{1+\nu}\left[\frac{1}{2}(g^{\alpha\omega}g^{\beta\rho}+g^{\omega\beta}g^{\alpha\rho})+\frac{\frac{\nu}{E}+\frac{\nu'^2}{E'}}{\frac{1-\nu}{E}-2\frac{\nu'^2}{E'}}g^{\omega\rho}g^{\alpha\beta}\right];$$

$$E^{\omega 3\theta 3} = G'g^{\omega\theta}, \qquad E^{\omega\rho 33} = \frac{\nu'}{\frac{1-\nu}{E}-2\frac{\nu'^2}{E'}}g^{\omega\rho}, \qquad E^{3333} = \frac{E'(1-\nu)}{E\left(\frac{1-\nu}{E}-2\frac{\nu'^2}{E'}\right)}$$

where E, ν are Young's modulus and Poisson's ratio, respectively, corresponding to the isotropy surface (the surface parallel at each point to x^3 = const.) and E', ν', G' are Young's modulus, Poisson's ratio and the shear modulus, respectively, corresponding to the plane normal to the isotropy surface.

We observe that the tensors E^{ijkl} and F_{ijmn} are spatial tensors, and by using the transformation relations between the spatial and surface components of a tensor [5], we obtain the expressions below, [3], which will be of use in our next considerations

$$E^{\alpha\beta\gamma\delta} = \overset{(0)}{E}{}^{\alpha\beta\gamma\delta}+z\overset{(1)}{E}{}^{\alpha\beta\gamma\delta}+\ldots+z^n\overset{(n)}{E}{}^{\alpha\beta\gamma\delta}+\ldots$$

$$E^{\alpha\beta 33} = \overset{(0)}{E}{}^{\alpha\beta 33}+z\overset{(1)}{E}{}^{\alpha\beta 33}+\ldots+z^n\overset{(n)}{E}{}^{\alpha\beta 33}+\ldots$$

$$E^{3333} = \overset{(0)}{E}{}^{3333}, \qquad F_{\alpha 3\omega 3} = \overset{(0)}{F}_{\alpha 3\omega 3}+z\overset{(1)}{F}_{\alpha 3\omega 3}+z^2\overset{(2)}{F}_{\alpha 3\omega 3}$$

where

$$\overset{(n)}{E}{}^{\alpha\beta\gamma\delta} = \overset{(n)}{E}{}^{\alpha\beta\gamma\delta}(\lambda^\omega), \qquad \overset{(n)}{E}{}^{\alpha\beta 33} = \overset{(n)}{E}{}^{\alpha\beta 33}(\lambda^\omega), \qquad \overset{(n)}{F}_{\alpha 3\omega 3} = \overset{(n)}{F}_{\alpha 3\omega 3}(\lambda^\theta).$$

In the following, the considered anisotropy will be of the type of elastic symmetry, with respect to the surface x^3 = const.† Accordingly, the physical equations are given by

$$\tau^{\omega\pi} = \tilde{E}^{\omega\pi\sigma\rho}e_{\sigma\rho}+\frac{E^{\omega\pi 33}}{E^{3333}}\tau^{33},$$

$$e_{33} = \frac{1}{E^{3333}}(\tau^{33}-E^{33\sigma\rho}e_{\sigma\rho}), \qquad e_{\lambda 3} = 2F_{\lambda 3\omega 3}\tau^{\omega 3},$$

where

$$\tilde{E}^{\omega\pi\sigma\rho} = E^{\omega\pi\sigma\rho}-\frac{E^{\omega\pi 33}E^{33\sigma\rho}}{E^{3333}}.$$

† For the remaining cases of anisotropy, the results can be deduced, obviously, by particularizing the relations obtained for this type of anisotropy.

3. EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

The equilibrium equations and the boundary conditions of the shell will be deduced by using a variational principle of the three-dimensional elastostatics,† which may be formulated as follows:

From all the possible states of stress, the only one which takes place, assigns to the functional

$$J = \int_\tau \rho \mathbf{H}\mathbf{V}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 + \int_{\Omega(\mathbf{P})} \mathbf{P}\mathbf{V}\,\mathrm{d}\Omega + \int_{\Omega(\mathbf{V})} \tau^{ik}\mathbf{g}_k m_i(\mathbf{V}^* - \mathbf{V})\,\mathrm{d}\Omega$$
$$- \int_\tau \{W - \tau^{ik}[e_{ik} - \tfrac{1}{2}(V_{k\|i} + V_{i\|k})]\}(\sqrt{g})\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \qquad (3.1)‡$$

a stationary value.

In (3.1) the following notations were used: $\mathbf{V}$ is the displacement vector, W the elastic potential per unit volume of the unstrained body; Ω the boundary of the body of volume τ; $\Omega_{(\mathbf{V})}$ the portion of the boundary Ω on which the displacement vector $\mathbf{V}^*$ is prescribed; $\Omega_{(\mathbf{P})}$ the portion of the boundary Ω on which the surface force vector $\mathbf{P}$ is prescribed; m_i the components of the unit vector of the internal normal to the boundary; $\mathbf{H} = (\sqrt{g})\mathbf{F}$ where $\mathbf{F} = F^i\mathbf{g}_i$ is the body force vector per unit mass; $g = |g_{ik}|$, ρ the density.

In the case of the shell, assuming $\delta\mathbf{H} = \delta\mathbf{P}$, we obtain from (3.1) [15],

$$\delta J = \int_\sigma \frac{1}{\sqrt{a}} \int_{-h/2}^{h/2} (\mathbf{T}_{i,i} + \rho\mathbf{H})\,\delta\mathbf{V}\,\mathrm{d}\sigma\,\mathrm{d}z + \int_{\Omega(\mathbf{P})} \left(\mathbf{T}_i\frac{n_i}{\sqrt{g}} + \mathbf{P}\right)\delta\mathbf{V}\,\mathrm{d}\Omega$$
$$+ \int_{\Omega(\mathbf{V})} (\mathbf{V}^* - \mathbf{V})\,\delta(\tau^{ik}\mathbf{g}_k m_i)\,\mathrm{d}\Omega + \int_\sigma \int_{-h/2}^{h/2} (\tau^{ik} - E^{ikmn}e_{mn})\left(\sqrt{\frac{g}{a}}\right)\delta e_{ik}\,\mathrm{d}\sigma\,\mathrm{d}z \qquad (3.2)$$
$$+ \int_\sigma \int_{-h/2}^{h/2} \left[e_{ik} - \frac{1}{2}(V_{i\|k} + V_{k\|i})\right]\left(\sqrt{\frac{g}{a}}\right)\delta\tau^{ik}\,\mathrm{d}\sigma\,\mathrm{d}z = 0$$

where $\mathrm{d}\sigma = (\sqrt{a})\,\mathrm{d}x^1\,\mathrm{d}x^2$ is the element of area of the middle surface;

$$\mathbf{T}_i = (\sqrt{g})\tau^{ik}\mathbf{g}_k = (\sqrt{a})(\lambda\lambda_\alpha^\rho\tau^{i\alpha}\mathbf{a}_\rho + \tau^{i3}\mathbf{n})$$

the stress vector [6, 16, 17];

$$\lambda_\alpha^\rho = \delta_\alpha^\rho - zb_\alpha^\rho; \qquad \lambda = \sqrt{\frac{g}{a}}. \qquad (3.3)$$

Considering that the variations of the displacements, stresses and strains in the interior and on the boundary of the shell are taken independently and assuming

$$\mathbf{V} = \sum_{i=0}^{\infty} \left(\overset{(i)}{V}_\beta\mathbf{a}^\beta + \overset{(i)}{V}_3\mathbf{n}\right) z^i \qquad (3.4)§$$

† In papers [7, 8] the Lagrange variational principle is used in the deduction of the equilibrium equations and boundary conditions for plates and shells of moderate thickness.

‡ This variational principle [9–11] constitutes a generalization of that due to Reissner [12]. For the three-dimensional elastodynamics a generalized Hamilton principle is given in [13]. (See also [14].)

§ See e.g. [7, 8, 17–20].

from the condition $\delta J = 0$, we obtain:

(i) the equilibrium equations

$$\int_{-h/2}^{h/2} (\mathbf{T}_{i,i}+\rho\mathbf{H})z^n \,\mathrm{d}z = 0, \qquad (n = 0, 1, \ldots) \tag{3.5}$$ †

(ii) the natural boundary conditions on $\Omega_{(\mathbf{P})}$

$$\int_{\Omega(\mathbf{P})} \left(\mathbf{T}_i\frac{m_i}{\sqrt{g}}+\mathbf{P}\right) \delta\mathbf{V}\,\mathrm{d}\Omega = 0 \tag{3.6}$$

(iii) the natural boundary conditions on $\Omega_{(\mathbf{V})}$

$$\int_{\Omega(\mathbf{V})} (\mathbf{V}^*-\mathbf{V})\,\delta(\tau^{ik}m_i\mathbf{g}_k)\,\mathrm{d}\Omega = 0. \tag{3.7}$$

Taking into account (3.3), equations (3.5) may be written in the form [3, 4]

$$L^{\omega\rho}_{(n)|\omega}+(n-1)b^{\rho}_{\omega}N^{\omega}_{(n)}-nN^{\rho}_{(n-1)}+[z^n\lambda\lambda^{\rho}_{\alpha}\tau^{3\alpha}]^{h/2}_{-h/2}+\mathscr{F}^{\rho}_{(n)} = 0 \tag{3.8}$$

$$N^{\alpha}_{(n)|\alpha}-nQ^{3}_{(n-1)}+b_{\rho\alpha}L^{\alpha\rho}_{(n)}+[z^n\lambda\tau^{33}]^{h/2}_{-h/2}+\mathscr{F}^{3}_{(n)} = 0 \qquad (n = 0, 1, \ldots). \tag{*}$$

$$L^{\omega\pi}_{(n)} = \int_{-h/2}^{h/2} \lambda\lambda^{\omega}_{\theta}\tau^{\theta\pi}z^n\,\mathrm{d}z, \qquad N^{\omega}_{(n)} = \int_{-h/2}^{h/2} \lambda\tau^{\omega 3}z^n\,\mathrm{d}z,$$

$$Q^{3}_{(n)} = \int_{-h/2}^{h/2} \lambda\tau^{33}z^n\,\mathrm{d}z, \tag{3.9}$$ ‡

$$\mathscr{F}^{\omega}_{(n)} = \int_{-h/2}^{h/2} \rho\lambda\lambda^{\omega}_{\alpha}F^{\alpha}z^n\,\mathrm{d}z, \qquad \mathscr{F}^{3}_{(n)} = \int_{-h/2}^{h/2} \rho\lambda F^{3}z^n\,\mathrm{d}z$$

which define the tensor components of the nth-order stress and body force couples respectively.

From (3.6) we obtain:

(a) The conditions on the bounding surfaces $s^{\pm}$ of the shell [3, 4]

$$[\lambda\tau^{3\alpha}z^n]^{h/2}_{-h/2} = p^{\alpha}_{(n)}, \qquad [\lambda\tau^{33}z^n]^{h/2}_{-h/2} = p^{3}_{(n)} \qquad (n = 0, 1, \ldots) \tag{3.10}$$

where we have used the notations

$$[\lambda P^{\alpha}z^n]^{h/2}_{-h/2} = p^{\alpha}_{(n)}, \qquad [\lambda P^{3}z^n]^{h/2}_{-h/2} = p^{3}_{(n)} \tag{3.11}$$

For $n = 2t+1$, $(t = 0, 1, \ldots)$

$$p^{\rho}_{(2t+1)} = \left(\frac{h}{2}\right)^{2t} p^{\rho}_{(1)}, \qquad p^{3}_{(2t+1)} = \left(\frac{h}{2}\right)^{2t} p^{3}_{(1)}$$

and for $n = 2t$, $(t = 0, 1, \ldots)$

$$p^{\rho}_{(2t)} = \left(\frac{h}{2}\right)^{2t} p^{\rho}_{(0)}, \qquad p^{3}_{(2t)} = \left(\frac{h}{2}\right)^{2t} p^{3}_{(0)}.$$

† Besides, the nth-order moment of the three-dimensional equilibrium equations (3.5) from $\delta J = 0$ the nth-order moments of the physical and of the geometrical equations corresponding to the three-dimensional elasticity also result. We shall not use the moments of these last equations but the equations themselves. A theory of plates based on the consideration of the nth-order moments of the fundamental equations of the three-dimensional elasticity is developed in [21, 22].

‡ Equations (3.8) may also represent the motion equations of the three-dimensional element of the shell through the replacing in $(3.9)_3$ of the components F^i by F^i-f^i [24], where f^i is the acceleration vector. In the following we consider $F^i = f^i = 0$.

(b) The natural static boundary conditions on $C_{(\mathbf{P})}$

$$\begin{gathered}
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}[P^{\alpha}+\tau^{\pi\alpha}v_{\pi}]\tau_{\beta}\lambda_{\alpha}^{\beta}z^{n}\,\mathrm{d}z = 0, \\
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}[P^{\alpha}+\tau^{\pi\alpha}v_{\pi}]v_{\beta}\lambda_{\alpha}^{\beta}z^{n}\,\mathrm{d}z = 0, \\
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}[P^{3}+\tau^{3\lambda}v_{\lambda}]z^{n}\,\mathrm{d}z = 0 \qquad (n = 0, 1, \ldots).
\end{gathered} \tag{3.12}$$

(c) The natural geometric boundary conditions on $C_{(\mathbf{V})}$

$$\begin{gathered}
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}(V_{\alpha}^{*}-V_{\alpha})v^{\alpha}z^{n}\,\mathrm{d}z = 0, \\
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}(V_{\alpha}^{*}-V_{\alpha})\tau^{\alpha}z^{n}\,\mathrm{d}z = 0, \\
\int_{-h/2}^{h/2} \sqrt{(g_{\lambda\varphi}\dot{x}^{\lambda}\dot{x}^{\varphi})}(V_{3}^{*}-V_{3})z^{n}\,\mathrm{d}z = 0 \qquad (n = 0, 1, \ldots)
\end{gathered} \tag{3.13}$$

where $\boldsymbol{\tau} = \tau_{\alpha}\mathbf{a}^{\alpha}$ and $\mathbf{v} = v_{\alpha}\mathbf{a}^{\alpha}$ are the unit vector tangent and normal respectively to the smooth curve C resulting from the intersection of the surfaces Σ and s, $(z = 0)$, $x^{\alpha} = x^{\alpha}(u)$ are the parametric equations of the curve C, $\dot{x}^{\alpha} = \mathrm{d}x^{\alpha}/\mathrm{d}u$.

4. EXPRESSIONS FOR STRAINS AND DISPLACEMENTS

On the basis of the strain–displacement linear equations

$$e_{ik} = \tfrac{1}{2}(V_{i\|k}+V_{k\|i}) \tag{4.1}$$

and using the physical equations $(2.10)_2$, $(2.10)_3$ and relations $(1.5)_2$, $(1.5)_3$, we deduce the equations [3, 4]

$$\begin{aligned}
V_{\alpha,3} &= 4F_{\alpha 3\lambda 3}\tau^{\lambda 3}-V_{3,\alpha}-2V_{\omega}[b_{\alpha}^{\omega}+zc_{\alpha}^{\omega}+z^{2}b_{\alpha}^{\gamma}c_{\gamma}^{\omega}+\ldots+z^{n}(\overset{(n)}{g}{}^{\omega\lambda}b_{\alpha\lambda}-\overset{(n-1)}{g}{}^{\omega\lambda}c_{\alpha\lambda})+\ldots], \\
V_{3,3} &= \frac{1}{E^{3333}}(\tau^{33}-E^{33\sigma\rho}c_{\sigma\rho})
\end{aligned} \tag{4.2}$$

which permit one to obtain the components of the vector $V_{i}(x^{\alpha}, x^{3})$ [3, 4, 18],

$$\begin{aligned}
\overset{(1)}{V}_{\alpha} &= 4\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{\rho 3})-\overset{(0)}{V}_{3,\alpha}-2\overset{(0)}{V}_{\omega}b_{\alpha}^{\omega}, \\
2\overset{(2)}{V}_{\alpha} &= 4(\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(1)}{\Lambda}+\overset{(1)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda})(\lambda\overset{(0)}{\tau}{}^{\rho 3})+4\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda}(\lambda\overset{(1)}{\tau}{}^{\rho 3})-\overset{(1)}{V}_{3,\alpha}-2\overset{(1)}{V}_{\omega}b_{\alpha}^{\omega}-2\overset{(0)}{V}_{\omega}c_{\alpha}^{\omega},\ldots, \\
n\overset{(n)}{V}_{\alpha} &= 4(\overset{(n-1)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda}+\overset{(n-2)}{F}_{\alpha 3\rho 3}\overset{(1)}{\Lambda}+\ldots+\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(n-1)}{\Lambda})(\lambda\overset{(0)}{\tau}{}^{\rho 3}) \\
&\quad +4(\overset{(n-2)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda}+\overset{(n-3)}{F}_{\alpha 3\rho 3}\overset{(1)}{\Lambda}+\ldots+\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(n-2)}{\Lambda})(\lambda\overset{(1)}{\tau}{}^{\rho 3}) \\
&\quad +\ldots+4\overset{(0)}{F}_{\alpha 3\rho 3}\overset{(0)}{\Lambda}(\lambda\overset{(n-1)}{\tau}{}^{\rho 3})-\overset{(n-1)}{V}_{3,\alpha}-2\overset{(0)}{V}_{\omega}(\overset{(n-1)}{g}{}^{\omega\lambda}b_{\alpha\lambda}-\overset{(n-2)}{g}{}^{\omega\lambda}c_{\alpha\lambda}) \\
&\quad -2\overset{(1)}{V}_{\omega}(\overset{(n-2)}{g}{}^{\omega\lambda}b_{\alpha\lambda}-\overset{(n-3)}{g}{}^{\omega\lambda}c_{\alpha\lambda})-\ldots-2\overset{(n-1)}{V}_{\omega}b_{\alpha}^{\omega},
\end{aligned} \tag{4.3}$$

$$\overset{(1)}{V}_3 = \frac{1}{E^{3333}}[\Lambda(\lambda \overset{(0)}{\tau}{}^{33}) - \overset{(0)}{E}{}^{33\alpha\beta}\overset{(0)}{e}_{\alpha\beta}], \ldots,$$

$$n\overset{(n)}{V}_3 = \frac{1}{E^{3333}}[\overset{(n-1)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33}) + \overset{(n-2)}{\Lambda}(\lambda\overset{(1)}{\tau}{}^{33}) + \ldots + \overset{(0)}{\Lambda}(\lambda\overset{(n-1)}{\tau}{}^{33}) \qquad (4.3\text{ cont.})$$
$$- \overset{(0)}{E}{}^{33\alpha\beta}\overset{(n-1)}{e}_{\alpha\beta} - \overset{(1)}{E}{}^{33\alpha\beta}\overset{(n-2)}{e}_{\alpha\beta} - \ldots - \overset{(n-1)}{E}{}^{33\alpha\beta}\overset{(0)}{e}_{\alpha\beta}],$$

where $\overset{(i)}{e}_{\alpha\beta}$ is the coefficient of z^i of the expansion into series of $e_{\alpha\beta}$, determined from (4.1) and $(1.5)_1$

$$\overset{(0)}{e}_{\alpha\beta} = \tfrac{1}{2}(\overset{(0)}{V}_{\alpha|\beta} + \overset{(0)}{V}_{\beta|\alpha} - 2b_{\alpha\beta}\overset{(0)}{V}_3),$$
$$\overset{(1)}{e}_{\alpha\beta} = \tfrac{1}{2}(\overset{(1)}{V}_{\alpha|\beta} + \overset{(1)}{V}_{\beta|\alpha} - 2\overset{(1)}{V}_3 b_{\alpha\beta} + 2\overset{(0)}{V}_3 c_{\alpha\beta} + 2\overset{(0)}{V}_\omega b^{\omega}_{\alpha|\beta}),$$
$$\overset{(2)}{e}_{\alpha\beta} = \tfrac{1}{2}(\overset{(2)}{V}_{\alpha|\beta} + \overset{(2)}{V}_{\beta|\alpha} - 2\overset{(2)}{V}_3 b_{\alpha\beta} + 2\overset{(1)}{V}_3 c_{\alpha\beta} + 2\overset{(1)}{V}_\omega b^{\omega}_{\alpha|\beta} + 2\overset{(0)}{V}_\omega b^{\omega}_{\delta} b^{\delta}_{\alpha|\beta}), \ldots \qquad (4.4)$$
$$\overset{(n)}{e}_{\alpha\beta} = \tfrac{1}{2}\{\overset{(n)}{V}_{\alpha|\beta} + \overset{(n)}{V}_{\beta|\alpha} - 2\overset{(n)}{V}_3 b_{\alpha\beta} + 2\overset{(n-1)}{V}_3 c_{\alpha\beta} + 2\overset{(n-1)}{V}_\omega b^{\omega}_{\alpha|\beta}$$
$$+ 2\overset{(n-2)}{V}_\omega b^{\omega}_{\delta} b^{\delta}_{\alpha|\beta} + \ldots - 2\overset{(1)}{V}_\omega[\overset{(n-1)}{g}{}^{\omega\pi}\overset{(0)}{\Gamma}_{\alpha\beta,\pi} - \overset{(n-2)}{g}{}^{\omega\pi}(b_{\beta\pi|\alpha} + 2\overset{0}{\Gamma}{}^{\tau}_{\alpha\beta} b_{\tau\pi})$$
$$+ \overset{(n-3)}{g}{}^{\omega\pi}(b_{\rho\pi} b^{\rho}_{\alpha|\beta} + \overset{0}{\Gamma}{}^{\rho}_{\alpha\beta} c_{\rho\pi})] - 2\overset{(0)}{V}_\omega[\overset{(n)}{g}{}^{\omega\pi}\overset{0}{\Gamma}_{\alpha\beta,\pi} - \overset{(n-1)}{g}{}^{\omega\pi}(b_{\beta\pi|\alpha} + 2\overset{0}{\Gamma}{}^{\tau}_{\alpha\beta} b_{\tau\pi})$$
$$+ \overset{(n-2)}{g}{}^{\omega\pi}(b_{\rho\pi} b^{\rho}_{\alpha|\beta} + \overset{0}{\Gamma}{}^{\rho}_{\alpha\beta} c_{\rho\pi})]\}, \ldots$$

Taking into account relations (2.10) (2.9), the stress–strain relations may be written

$$\tau^{\omega\pi} = \overset{\widetilde{0}}{E}{}^{\omega\pi\alpha\beta}\overset{(0)}{e}_{\alpha\beta} + z(\overset{\widetilde{(0)}}{E}{}^{\omega\pi\alpha\beta}\overset{(1)}{e}_{\alpha\beta} + \overset{\widetilde{(1)}}{E}{}^{\omega\pi\alpha\beta}\overset{(0)}{e}_{\alpha\beta}) + \ldots$$
$$+ z^n(\overset{\widetilde{(0)}}{E}{}^{\omega\pi\alpha\beta}\overset{(n)}{e}_{\alpha\beta} + \overset{\widetilde{(1)}}{E}{}^{\omega\pi\alpha\beta}\overset{(n-1)}{e}_{\alpha\beta} + \ldots + \overset{\widetilde{(n)}}{E}{}^{\omega\pi\alpha\beta}\overset{(0)}{e}_{\alpha\beta}) + \frac{1}{E^{3333}}\{\overset{(0)}{E}{}^{\omega\pi33}\overset{(0)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33})$$
$$+ z[\overset{(0)}{E}{}^{\omega\pi33}\overset{(1)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33}) + \overset{(0)}{E}{}^{\omega\pi33}\overset{(0)}{\Lambda}(\lambda\overset{(1)}{\tau}{}^{33}) + \overset{(1)}{E}{}^{\omega\pi33}\overset{(0)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33})] + \ldots \qquad (4.5)$$
$$+ z^n\{\overset{(0)}{E}{}^{\omega\pi33}[\overset{(n)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33}) + \overset{(n-1)}{\Lambda}(\lambda\overset{(1)}{\tau}{}^{33}) + \ldots + \overset{(0)}{\Lambda}(\lambda\overset{(n)}{\tau}{}^{33})]$$
$$+ \overset{(1)}{E}{}^{\omega\pi33}[\overset{(n-1)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33}) + \ldots + \overset{(0)}{\Lambda}(\lambda\overset{(n-1)}{\tau}{}^{33})] + \ldots + \overset{(n)}{E}{}^{\omega\pi33}\overset{(0)}{\Lambda}(\lambda\overset{(0)}{\tau}{}^{33})\} + \ldots$$

where

$$\overset{\widetilde{(0)}}{E}{}^{\omega\pi\alpha\beta} = \overset{(0)}{E}{}^{\omega\pi\alpha\beta} - \frac{\overset{(0)}{E}{}^{\omega\pi33}\overset{(0)}{E}{}^{33\alpha\beta}}{E^{3333}},$$

$$\overset{\widetilde{(1)}}{E}{}^{\omega\pi\alpha\beta} = \overset{(1)}{E}{}^{\omega\pi\alpha\beta} - \frac{\overset{(1)}{E}{}^{\omega\pi33}\overset{(0)}{E}{}^{33\alpha\beta} + \overset{(0)}{E}{}^{\omega\pi33}\overset{(1)}{E}{}^{33\alpha\beta}}{E^{3333}}, \ldots, \qquad (4.6)$$

$$\overset{\widetilde{(n)}}{E}{}^{\omega\pi\alpha\beta} = \overset{(n)}{E}{}^{\omega\pi\alpha\beta} - \frac{\overset{(n)}{E}{}^{\omega\pi33}\overset{(0)}{E}{}^{33\alpha\beta} + \overset{(n-1)}{E}{}^{\omega\pi33}\overset{(1)}{E}{}^{33\alpha\beta} + \ldots + \overset{(0)}{E}{}^{\omega\pi33}\overset{(n)}{E}{}^{33\alpha\beta}}{E^{3333}}.$$

In relations (4.5), (4.3), $(\lambda \overset{(n)}{\tau}{}^{i3})$ represent the coefficients of z^n of the expansion into series of $\lambda\tau^{i3}$, [3, 4, 25]

$$\begin{aligned}\lambda\tau^{i3} = {} & \frac{p^i_{(1)}}{h}+\frac{p^i_{(0)}}{h}z+\overset{(2)}{\varphi}{}^i\left(z^2-\frac{h^2}{4}\right)+\overset{(3)}{\varphi}{}^i z\left(z^2-\frac{h^2}{4}\right)+\ldots+\overset{(2n)}{\varphi}{}^i\left(z^{2n}-\frac{h^{2n}}{2^{2n}}\right) \\ & +\overset{(2n+1)}{\varphi}{}^i z\left(z^{2n}-\frac{h^{2n}}{2^{2n}}\right)+\ldots\end{aligned} \tag{4.7}$$

and $\Lambda = 1/\lambda$.

The functions $\overset{(n)}{\varphi}{}^i = \overset{(n)}{\varphi}{}^i(x^\alpha)$, $(n = 2, 3, \ldots)$ together with the displacements of the middle surface $\overset{(0)}{V}_i(x^\alpha)$ constitute the basic unknowns of the problem. In the case of the Nth-order approximation of the development (4.7)† $(N = 2, 3, \ldots)$, by replacing the expression of the nth-order stress couples $(n = 0, 1, \ldots (N-1))$ in the equilibrium equations (3.8), we obtain a system of $3N$ partial differential equations for the $3N$ unknowns of the problems $\overset{(0)}{V}_\alpha$, $\overset{(0)}{V}_3$, $\overset{(2)}{\varphi}{}^i, \ldots \overset{(N)}{\varphi}{}^i$.

In all the previous developments no restriction concerning the shell thickness was made, a fact which makes possible a study of the problem of thick shells.

5. POSSIBILITY OF REDUCING THE NUMBER OF UNKNOWNS

In papers[3, 4] the system of equations

$$\begin{aligned} 2\overset{(2)}{\varphi}{}^3 &= -(\lambda\overset{(1)}{\tau}{}^{\alpha 3})_{|\alpha}-(\lambda\overset{(1)}{\tau}{}^{\alpha\varphi})b_{\alpha\varphi}+(\lambda\overset{(0)}{\tau}{}^{\alpha\varphi})c_{\alpha\varphi}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2n\overset{(2n)}{\varphi}{}^3 &= -(\lambda\overset{(2n-1)}{\tau}{}^{\alpha 3})_{|\alpha}-(\lambda\overset{(2n-1)}{\tau}{}^{\alpha\varphi})b_{\alpha\varphi}+(\lambda\overset{(2n-2)}{\tau}{}^{\alpha\varphi})c_{\alpha\varphi}, \\ (2n+1)\overset{(2n+1)}{\varphi}{}^3 &= -(\lambda\overset{(2n)}{\tau}{}^{\alpha 3})_{|\alpha}-(\lambda\overset{(2n)}{\tau}{}^{\alpha\varphi})b_{\alpha\varphi}+(\lambda\overset{(2n-1)}{\tau}{}^{\alpha\varphi})c_{\alpha\varphi}, \ldots \end{aligned} \tag{5.1}$$

was deduced, which permits the determination of the functions $\overset{(2)}{\varphi}{}^3, \ldots, \overset{(n)}{\varphi}{}^3, \ldots$ by means of the remaining unknowns of the problem. Since $(\lambda\overset{(i)}{\tau}{}^{\lambda\varphi})$ contains in general case $(\lambda\overset{(i)}{\tau}{}^{33})$ under differential form, [3], the solution of this system of equations is laborious. In some particular cases, however (e.g. in the case of the plane plate or when additional assumptions are introduced), the system (5.1) becomes an algebraic system of equations. Thus in the case of the Nth-order approximation of the expansion (4.7) $\overset{(2)}{\varphi}{}^3(x^\alpha), \ldots, \overset{(N+1)}{\varphi}{}^3(x^\alpha), \ldots$ may be expressed by the functions $\overset{(2)}{\varphi}{}^\rho(x^\alpha), \ldots, \overset{(N)}{\varphi}{}^\rho(x^\alpha), \ldots$ which together with $V_i(x^\alpha)$ constitute the basic unknowns of the problem. We obtain in this case a system of $(2N+1)$ equations with $(2N+1)$ unknowns, indicated above. We mentioned that the manner of determining the functions $\overset{(i)}{\varphi}{}^3(x^\alpha)$ results in the fact that from the system of equilibrium equations (3.8), the equations indicated by a star are identically satisfied, (for $n \geqslant 1$).

† In the case of the Nth-order approximation of (4.7), $\overset{(r)}{\varphi}{}^i = 0$ if $r > n$.

6. PLATE EQUATIONS

The consideration of the middle surface as a plane surface leads to certain simplifications.†

In this case the equations and the natural boundary conditions may be grouped as follows‡

(a)

$$
A_{(n)}\left\{
\begin{aligned}
& L^{\omega\rho}_{(n)|\omega} - nN^{\rho}_{(n-1)} + p^{\rho}_{(n)} = 0,\\
& \overset{(n)}{e}_{\alpha\beta} = \tfrac{1}{2}\big(\overset{(n)}{V}_{\alpha|\beta} + \overset{(n)}{V}_{\beta|\alpha}\big),\\
& \overset{(n)}{V}_{\alpha} = \frac{1}{n}\big(4F_{\alpha 3\rho 3}\overset{(n-1)}{\tau}{}^{\rho 3} - \overset{(n-1)}{V}_{3,\alpha}\big),\\
& L^{\omega}_{(n)} = \widetilde{E}^{\omega\pi\alpha\beta}\Bigg[\frac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\overset{(0)}{e}_{\alpha\beta} + \frac{h^{n+2}(1^{n+2}-(-1)^{n+2})}{(n+2)2^{n+2}}\overset{(0)}{e}_{\alpha\beta} + \dots\\
&\qquad + \frac{h^{n+p+1}(1^{n+p+1}-(-1)^{n+p+1})}{(n+p+1)2^{n+p+1}}\overset{(p)}{e}_{\alpha\beta} + \dots\Bigg]\\
&\qquad + \frac{E^{\omega\pi 33}}{E^{3333}}\Bigg[\frac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\overset{(0)}{\tau}{}^{33} + \dots\Bigg]\\
&\qquad + \frac{h^{n+r+1}(1^{n+r+1}-(-1)^{n+r+1})}{(n+r+1)2^{n+r+1}}\overset{(r)}{\tau}{}^{33} + \dots \qquad (p, r = 0, 1, \dots).
\end{aligned}
\right.
$$

$$
B_{(n)}\left\{
\begin{aligned}
& N^{\alpha}_{(n)|\alpha} - nQ^{3}_{(n-1)} + p^{3}_{(n)} = 0,\\
& \overset{(n)}{V}_{3} = \frac{1}{nE^{3333}}\big(\overset{(n-1)}{\tau}{}^{33} - E^{33\alpha\beta}\overset{(n-1)}{e}_{\alpha\beta}\big),\\
& N^{\alpha}_{(n)} = \frac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\frac{p^{\alpha}_{(1)}}{h} + \frac{h^{n+2}(1^{n+2}-(-1)^{n+2})}{(n+2)2^{n+2}}\frac{p^{\alpha}_{(0)}}{h}\\
&\qquad + \frac{h^{n+3}}{2^{n+3}}\left(\frac{1^{n+3}-(-1)^{n+3}}{n+3} - \frac{1^{n+1}-(-1)^{n+1}}{n+1}\right)\overset{(2)}{\varphi}{}^{\alpha}\\
&\qquad + \frac{h^{n+4}}{2^{n+4}}\left(\frac{1^{n+4}-(-1)^{n+4}}{n+4} - \frac{1^{n+2}-(-1)^{n+2}}{n+2}\right)\overset{(3)}{\varphi}{}^{\alpha} + \dots\\
&\qquad + \frac{h^{n+2p+1}}{2^{n+2p+1}}\left(\frac{1^{n+2p+1}-(-1)^{n+2p+1}}{n+2p+1} - \frac{1^{n+1}-(-1)^{n+1}}{n+1}\right)\overset{(2p)}{\varphi}{}^{\alpha}\\
&\qquad + \frac{h^{n+2p+2}}{2^{n+2p+2}}\left(\frac{1^{n+2p+2}-(-1)^{n+2p+2}}{n+2p+2} - \frac{1^{n+2}-(-1)^{n+2}}{n+2}\right)\overset{(2p+1)}{\varphi}{}^{\alpha} + \dots\\
&\qquad\qquad (p = 1, 2, \dots).
\end{aligned}
\right.
$$

† In this case we have $b_{\alpha\beta} = 0$ and hence $E^{ijkl} = \overset{(0)}{E}{}^{ijkl}$, $F_{ijmn} = \overset{(0)}{F}_{ijmn}$. Likewise, the order of the covariant differentiation with respect to the surface coordinates x^{α} is immaterial.

‡ The sense of this grouping of equations will appear as obvious in the following considerations.

(b) Natural static boundary conditions

$$A'_{(n)}\begin{cases} L^{\omega\rho}_{(n)}\nu_\rho\tau_\omega + \displaystyle\int_{-h/2}^{h/2} P^\alpha\tau_\alpha z^n\,\mathrm{d}z = 0,\\ L^{\omega\rho}_{(n)}\nu_\rho\nu_\omega + \displaystyle\int_{-h/2}^{h/2} P^\alpha\nu_\alpha z^n\,\mathrm{d}z = 0,\end{cases}$$

$$B'_{(n)}\Big\{\quad N^\lambda_{(n)}\nu_\lambda + \int_{-h/2}^{h/2} P^3 z^n\,\mathrm{d}z = 0.$$

(c) Natural geometric boundary conditions

$$A''_{(n)}\begin{cases}\left[\dfrac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\overset{(0)}{V}_\alpha + \ldots + \dfrac{h^{n+p+1}(1^{n+p+1}-(-1)^{n+p+1})}{(n+p+1)2^{n+p+1}}\overset{(p)}{V}_\alpha + \ldots\right]\nu^\alpha\\ \qquad = \displaystyle\int_{-h/2}^{h/2} V^*_\alpha \nu^\alpha z^n\,\mathrm{d}z,\\ \left[\dfrac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\overset{(0)}{V}_\alpha + \ldots + \dfrac{h^{n+p+1}(1^{n+p+1}-(-1)^{n+p+1})}{(n+p+1)2^{n+p+1}}\overset{(p)}{V}_\alpha + \ldots\right]\tau^\alpha\\ \qquad = \displaystyle\int_{-h/2}^{h/2} V^*_\alpha \tau^\alpha z^n\,\mathrm{d}z.\end{cases}$$

$$B''_{(n)}\begin{cases}\dfrac{h^{n+1}(1^{n+1}-(-1)^{n+1})}{(n+1)2^{n+1}}\overset{(0)}{V}_3 + \ldots + \dfrac{h^{n+r+1}(1^{n+r+1}-(-1)^{n+r+1})}{(n+r+1)2^{n+r+1}}\overset{(r)}{V}_3\\ \qquad = \displaystyle\int_{-h/2}^{h/2} V^*_3 z^n\,\mathrm{d}z \qquad (p, r = 0, 1, \ldots).\end{cases}$$

The sets of equations (A_n), (B_n), together with the boundary conditions (A'_n), (B'_n) or (A''_n), (B''_n), fall into two independent groups, namely (A_{2q}), (B_{2q+1}) to which we associate the boundary conditions (A'_{2q}), (B'_{2q+1}) or (A''_{2q}), (B''_{2q+1}), $(q = 0, 1, \ldots)$ and respectively (A_{2q+1}), (B_{2q}) with the boundary conditions (A'_{2q+1}), (B'_{2q}) or (A''_{2q+1}), (B''_{2q}), $(q = 0, 1, \ldots)$. The first group of equations corresponds to the state of stress of the type known as generalized plane stress and the latter one to transverse bending of plates. In view of their complexity, these groups of equations will not be written explicitly here.

7. THE EQUATIONS OF PLATES OF MODERATE THICKNESS

The equations for plates of moderate thickness will be deduced by considering simultaneously [7, 8, 26–31]

$$\frac{kf_1^2h^2}{L^2} < 1, \qquad \frac{f_2 f_1^4 h^4}{L^4} \ll 1 \tag{7.1}$$

where L is a characteristic dimension of the mid-plane, f_1 the variation index of the state of stress [32], k and f_2 are physical factors defined respectively by the maximum of ratios,

$$k;\ \frac{c^{\omega\rho}_{\lambda\varphi}}{c^{33}_{33}},\ \frac{c^{33}_{\lambda\varphi}}{c^{33}_{33}},\ c^{\omega\rho}_{\alpha\beta}s^{\pi 3}_{\sigma 3},\ c^{33}_{\alpha\beta}s^{\pi 3}_{\sigma 3}$$

$$f_2;\ \frac{c^{\omega\rho}_{\alpha\varphi}c^{33}_{\sigma\pi}}{c^{33}_{33}}s^{\omega 3}_{\gamma 3},\ \frac{c^{33}_{x\varphi}c^{33}_{\pi\sigma}}{c^{33}_{33}}s^{\omega 3}_{\gamma 3},\ c^{\lambda\varphi}_{\pi\sigma}c^{\omega\rho}_{\alpha\beta}s^{\theta 3}_{\mu 3}s^{\pi 3}_{\tau 3}, \tag{7.2}$$

$$c^{33}_{\pi\sigma}c^{33}_{\alpha\beta}s^{\theta 3}_{\mu 3}s^{\pi 3}_{\tau 3},\ \frac{c^{\omega\rho}_{\lambda\varphi}c^{\pi\sigma}_{\alpha\beta}}{(c^{33}_{33})^2},\ \frac{c^{33}_{\lambda\varphi}c^{\pi\sigma}_{33}}{(c^{33}_{33})^2}.$$

where c^{ij}_{mn} and s^{ij}_{mn} are the elastic coefficients in Cartesian rectangular coordinates, corresponding to E^{ij}_{mn} and F^{ij}_{mn}, respectively.

In this case it is sufficient to consider the third order approximation of the expansion (4.7),

$$\tau^{\rho 3} = \frac{p^{\rho}_{(1)}}{h} + p^{\rho}_{(0)}\frac{z}{h} + \overset{(2)}{\varphi}{}^{\rho}\left(z^2 - \frac{h^2}{4}\right) + \overset{(3)}{\varphi}{}^{\rho} z\left(z^2 - \frac{h^2}{4}\right). \tag{7.3}$$

From system (5.1), taking into consideration (7.3), we obtain for

$$\tau^{33} = \overset{(0)}{\tau}{}^{33} + z\overset{(1)}{\tau}{}^{33} + z^2\overset{(2)}{\tau}{}^{33} + z^3\overset{(3)}{\tau}{}^{33} + z^4\overset{(4)}{\tau}{}^{33}. \tag{7.4}$$

where

$$\begin{aligned}
\overset{(0)}{\tau}{}^{33} &= \frac{p^3_{(1)}}{h} + \frac{h}{8}p^{\rho}_{(0)|\rho} - \frac{h^4}{64}\overset{(3)}{\varphi}{}^{\rho}|_{\rho}, \\
\overset{(1)}{\tau}{}^{33} &= -\frac{1}{h}p^{\rho}_{(1)|\rho} + \frac{h^2}{4}\overset{(2)}{\varphi}{}^{\rho}|_{\rho}, \\
\overset{(2)}{\tau}{}^{33} &= -\frac{1}{2h}p^{\rho}_{(0)|\rho} + \frac{h^2}{8}\overset{(3)}{\varphi}{}^{\rho}|_{\rho}, \\
\overset{(3)}{\tau}{}^{33} &= -\tfrac{1}{3}\overset{(2)}{\varphi}{}^{\rho}|_{\rho}, \qquad \overset{(4)}{\tau}{}^{33} = -\tfrac{1}{4}\overset{(3)}{\varphi}{}^{\rho}|_{\rho}.
\end{aligned} \tag{7.5}$$

8. EQUATIONS FOR THE TRANSVERSE BENDING OF PLATES OF MODERATE THICKNESS

Taking into account the groups of equations (A_{2r+1}), (B_{2r}) (Section 6), and using relations (7.1)–(7.5), we deduce the expressions for τ^{ij} corresponding to this state of stress.

$$\begin{aligned}
\tau^{\rho 3} &= \frac{p^{\rho}_{(1)}}{h} + \overset{(2)}{\varphi}{}^{\rho}\left(z^2 - \frac{h^2}{4}\right), \\
\tau^{33} &= z\overset{(1)}{\tau}{}^{33} + z^3\overset{(3)}{\tau}{}^{33}, \\
\tau^{\omega\pi} &= \tilde{E}^{\omega\pi\alpha\beta}(z\overset{(1)}{e}_{\alpha\beta} + z^3\overset{(3)}{e}_{\alpha\beta}) + \frac{E^{\omega\pi 33}}{E^{3333}}(z\overset{(1)}{\tau}{}^{33} + z^3\overset{(3)}{\tau}{}^{33}).
\end{aligned} \tag{8.1}$$

The unknown functions are $\overset{(0)}{V}_3$ and $\overset{(2)}{\varphi}{}^{\rho}$.

Taking into account the relations given in Sections 6 and 9 (see also [3, 4]) by satisfying the equilibrium equations

$$\begin{aligned}
L^{\omega\rho}_{(1)|\omega} - N^{\rho}_{(0)} + p^{\rho}_{(1)} &= 0, \\
N^{\omega}_{(0)|\omega} + p^3_{(0)} &= 0
\end{aligned} \tag{8.2}$$

we obtain the following governing partial differential equations in $\overset{(0)}{V}_3$ and $\overset{(2)}{\varphi}{}^\rho$

$$
\begin{aligned}
&\tilde{E}^{\omega\rho\alpha\beta}\left(\overset{(0)}{V}_{3|\alpha\beta\omega}+\frac{h^2}{40}\frac{E^{33\lambda\varphi}}{E^{3333}}\overset{(0)}{V}_{3|\lambda\varphi\alpha\beta\omega}\right)\\
&\qquad+\frac{2}{5}h^2\tilde{E}^{\omega\rho\alpha\beta}(F_{\alpha3\lambda3}\overset{(2)}{\varphi}{}^\lambda_{|\beta\omega}+F_{\beta3\lambda3}\overset{(2)}{\varphi}{}^\lambda_{|\alpha\omega})\\
&\qquad-\frac{1}{5}h^2\frac{E^{\omega\rho33}}{E^{3333}}\overset{(2)}{\varphi}{}^\sigma_{|\sigma\omega}-\frac{2}{h}\tilde{E}^{\omega\rho\alpha\beta}(F_{\alpha3\lambda3}p^\lambda_{(1)|\beta\omega}+F_{\beta3\lambda3}p^\lambda_{(1)|\alpha\omega})\\
&\qquad+\frac{1}{h}\frac{E^{\omega\rho33}}{E^{3333}}p^\gamma_{(1)|\gamma\omega}-2\overset{(2)}{\varphi}{}^\rho=0,\qquad \frac{h^3}{6}\overset{(2)}{\varphi}{}^\alpha_{|\alpha}-p^3_{(0)}-p^\alpha_{(1)|\alpha}=0.
\end{aligned}
\tag{8.3}
$$

The sixth order of this system of equations requires three boundary conditions on each edge. These static and geometric natural boundary conditions may easily be obtained from the corresponding expressions contained in Section 6, and are given by (A_1'), (B_0') and (A_1''), (B_0''), $(p=3, r=2)$, respectively.

9. STRETCHING EQUATIONS FOR PLATES OF MODERATE THICKNESS-GENERALIZED PLANE STRESS

Taking into account the groups of equations (A_{2r}), (B_{2r+1}) (Section 6) and relations (7.1)–(7.4) we deduce the expressions of τ^{ij} corresponding to this state of stress

$$
\begin{aligned}
\tau^{\omega3}&=p^\omega_{(0)}\frac{z}{h}+\overset{(3)}{\varphi}{}^\omega z\left(z^2-\frac{h^2}{4}\right),\\
\tau^{33}&=\overset{(0)}{\tau}{}^{33}+z^2\overset{(2)}{\tau}{}^{33}+z^4\overset{(4)}{\tau}{}^{33},\\
\tau^{\omega\pi}&=\tilde{E}^{\omega\pi\alpha\beta}(\overset{(0)}{e}_{\alpha\beta}+z^2\overset{(2)}{e}_{\alpha\beta}+z^4\overset{(4)}{e}_{\alpha\beta})\\
&\qquad+\frac{E^{\omega\pi33}}{E^{3333}}(\overset{(0)}{\tau}{}^{33}+z^2\overset{(2)}{\tau}{}^{33}+z^4\overset{(4)}{\tau}{}^{33}).
\end{aligned}
\tag{9.1}
$$

The unknowns are $\overset{(0)}{V}_\alpha$ and $\overset{(3)}{\varphi}{}^\rho$

Taking into account the relations given in Sections 6 and 7 (see also [3, 4]) by satisfying the equilibrium equations

$$
\begin{aligned}
L^{\omega\rho}_{(0)|\omega}+p^\rho_{(0)}&=0,\\
L^{\omega\rho}_{(2)|\omega}-2\dot{N}^\rho_{(1)}+\left(\frac{h}{2}\right)^2p^\rho_{(0)}&=0,
\end{aligned}
$$

the following system of equations in $\overset{(0)}{V}_\alpha$ and $\overset{(3)}{\varphi}{}^\rho$ are obtained

$$
\begin{aligned}
&\tilde{E}^{\omega\pi\alpha\beta}\left\{\frac{h}{2}(\overset{(0)}{V}_{\alpha|\beta\omega}+\overset{(0)}{V}_{\beta|\alpha\omega})+\frac{h^3}{48}\frac{E^{33\lambda\varphi}}{E^{3333}}(\overset{(0)}{V}_{\lambda|\varphi\alpha\beta\omega}+\overset{(0)}{V}_{\varphi|\lambda\alpha\beta\omega})\right.\\
&\qquad+\frac{h^7}{2^7\times15E^{3333}}\overset{(3)}{\varphi}{}^\rho_{|\rho\alpha\beta\omega}+\frac{7h^5}{2^5\times15}(F_{\alpha3\rho3}\overset{(3)}{\varphi}{}^\rho_{|\beta\omega}+F_{\beta3\rho3}\overset{(3)}{\varphi}{}^\rho_{|\alpha\omega})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{h^7}{2^8\times 15}\frac{E^{33\lambda\varphi}}{E^{3333}}(F_{\lambda 3\rho 3}\overset{(3)}{\varphi}{}^{\rho}|_{\varphi\alpha\beta\omega}+F_{\varphi 3\rho 3}\overset{(3)}{\varphi}{}^{\rho}|_{\lambda\alpha\beta\omega})\\
&\left.+\frac{h^2}{12}(p^{\rho}_{(0)|\beta\omega}F_{\alpha 3\rho 3}+p^{\rho}_{(0)|\alpha\omega}F_{\beta 3\rho 3})-\frac{h^2}{24E^{3333}}\overset{3}{p}_{(1)|\alpha\beta\omega}\right\}\\
&+\frac{E^{\omega\pi 33}}{E^{3333}}\left(\overset{3}{p}_{(1)|\omega}+\frac{h^2}{12}p^{\rho}_{(0)|\rho\omega}-\frac{h^5}{120}\overset{(3)}{\varphi}{}^{\rho}|_{\rho\omega}\right)+p^{\pi}_{(0)}=0,
\end{aligned}
\tag{9.2}
$$

$$
\begin{aligned}
\tilde{E}^{\omega\pi\alpha\beta}&\left\{\frac{h^3}{24}(\overset{(0)}{V}_{\alpha|\beta\omega}+\overset{(0)}{V}_{\beta|\alpha\omega})-\frac{h^5}{320}\frac{E^{33\lambda\varphi}}{E^{3333}}(\overset{(0)}{V}_{\lambda|\varphi\alpha\beta\omega}+\overset{(0)}{V}_{\varphi|\lambda\alpha\beta\omega})\right.\\
&-\frac{9h^7}{2^7\times 35}(F_{\alpha 3\rho 3}\overset{(3)}{\varphi}{}^{\rho}|_{\beta\omega}+F_{\beta 3\rho 3}\overset{(3)}{\varphi}{}^{\rho}|_{\alpha\omega})\\
&\left.+\frac{h^4}{80}(F_{\alpha 3\rho 3}p^{\rho}_{(0)|\beta\omega}+F_{\beta 3\rho 3}p^{\rho}_{(0)|\alpha\omega})-\frac{h^4}{160E^{3333}}\overset{3}{p}_{(1)|\alpha\beta\omega}\right\}\\
&+\frac{E^{\omega\pi 33}}{E^{3333}}\frac{h^2}{12}(\overset{3}{p}_{(1)|\omega}+\frac{h^2}{20}p^{\rho}_{(0)|\rho\omega}-\frac{h^5}{2^3\times 35}\overset{(3)}{\varphi}{}^{\rho}|_{\rho\omega})\\
&+\frac{h^2}{12}p^{\pi}_{(0)}+\frac{h^5}{60}\overset{(3)}{\varphi}{}^{\pi}=0.
\end{aligned}
$$

The static or geometric natural boundary conditions may be obtained by using expressions (A_0'), (B_1') or (A_0''), (B_1''), $(p = 4, r = 3)$ respectively, which were given in Section 6.

10. THE TRANSVERSE BENDING FOR RECTANGULAR PLATES OF MODERATE THICKNESS

As an example we consider the transverse bending of rectangular plates made of a transversely isotropic material.

The plate edges ($\alpha = 0, a; \beta = 0, b$) are assumed to be simply supported and hence along them, the moments and the deflection are assumed to vanish.

We consider that the loads are distributed according to the law

$$
\overset{3}{p}_{(0)} = p^{*}_{(0)}\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{a}
\tag{10.1}
$$

The expressions

$$
\begin{aligned}
&\overset{(2)}{\varphi}_1 = B\cos\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}, \qquad \overset{(2)}{\varphi}_2 = C\sin\frac{\pi\alpha}{a}\cos\frac{\pi\beta}{b},\\
&\overset{(0)}{V}_3 = A\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}
\end{aligned}
\tag{10.2}
$$

satisfy all the conditions on the plate edges [27].

Taking into account relations (2.2), (10.1), (10.2) in equations (8.3), we deduce the following expression for the deflection of the plate center

$$\overset{(0)}{V}_3 = (\overset{(0)}{V}_3)_N \frac{1+\dfrac{\pi^2h^2(1+\varphi^2)}{10a^2}\left(\dfrac{E(1-\nu/2)}{G'(1-\nu^2)}-\dfrac{\nu'E}{E'(1-\nu)}\right)}{1-\dfrac{\pi^2h^2(1+\varphi^2)}{40a^2}\dfrac{E\nu'}{E'(1-\nu)}} \tag{10.3}$$

where

$$(\overset{(0)}{V}_3)_N = \frac{12p^*_{(0)}a^4(1-\nu^2)}{E\pi^4h^3(1+\varphi^2)^2}, \qquad \varphi = \frac{a}{b}. \tag{10.4}$$

In the following, for the case of the isotropic quadratic plate ($h/a = \frac{1}{3}$; $\nu = 0{,}3$; $\varphi = 1$) we shall compare the expression of the deflection of the plate centre, obtained by various theories:

	The exact theory [33]	The suggested theory	The classical theory
$(\overset{(0)}{V}_3)_c = \overset{(0)}{V}_3\dfrac{E}{p^*_{(0)}h}$	3·492	3·35	2·27

Results concerning the transverse deflection obtained in this example, show, on one hand a good correlation with values obtained within the exact theory (the deviation being of 4·25 per cent) while on the other hand a difference of 32·2 per cent is obtained, with respect to the classical theory.

By contrast with this case, the relation (10.3) permits one to infer readily (it should be remarked however, that these conclusions are general in character being not limited only to the case of anisotropic plane plates) that even for "geometrically thin" anisotropic plates ($h^2/L^2 \ll 1$), a more refined theory might lead to results quite different from those obtained within the classical theory, if the material possesses a high degree of anisotropy† (the anisotropy degree is defined by the maximum value of the ratios $(7.2)_1$)

11. DISCUSSION

The present paper develops a theory of elastic anisotropic shells and plates, the Love–Kirchhoff hypothesis being abrogated. Implicitly the contradictions introduced by this hypothesis are eliminated. No restrictions are made concerning the thickness of the shell or plate. The theory is approached within a general framework so that, by particularization, one can obtain a series of results previously deduced.

(i) Thus: the equation given in Sections 4 and 8 contain the results deduced by Ambartsumian for anisotropic shells of moderate thickness in the case $e_{33} = \tau^{33} = 0$ [28, 29] and for plates of moderate thickness in the case $e_{33} = 0$ [27, 29, 30] as well as those deduced by Mushtari and by Teregulov [7, 8] for the case of isotropic plates of moderate thickness.

Also, our results are in agreement with those deduced by Vekua [23] for thin shallow

† This conclusion was first made evident by Ambartsumian [28–31].

isotropic shells, or with the linearized version of Habip's equations [14], deduced for anisotropic shells, in the case of the linear variation of the displacements through the thickness.

(ii) The theory presented in [26, 27–29, 34, 35] leads, within the frame of our example (Section 10), to values of the deflection $(\overset{(0)}{V}_3)_C$ larger than that obtained through the exact theory [33]; in contrast with these theories, the proposed theory† leads to a value of $(\overset{(0)}{V}_3)_C$ which is close to the exact one, but inside the domain between the classic and the exact theory.

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† This is also valid for the theory given by Reissner [36].

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Résumé—Cette étude présente une théorie linéaire pour les plaques et les coques homogènes, anisotropiques et élastiques, établie sans prendre en considération les hypothèses de Love–Kirchhoff.

Les conditions aux limites sur les surfaces exterieures de la coque sont rigoureusement satisfaites. Aucune restriction n'est portée sur l'épaisseur de la coque ce qui permet une étude des plaques et discoques épaisses. Pour conclure, le problème des plaques anisotropiques élastiques et examiné à l'aide des résultats obtenus dans la première partie.

Zusammenfassung—Diese Arbeit behandelt die lineare Theorie anisotropischer elastischer Platten und Schalen, ohne die Love–Kirchhoff'schen Voraussetzungen zu berücksichtigen.

Die Randwertsbedingungen der äusseren Grenzen werden erfüllt. Keinerlei Einschränkungen der Schalendicke werden gemacht, dies ermöglicht die Untersuchung dicker Schalen und Wände. Schliesslich werden mit Hilfe der Resultate, die im ersten Teil erzielt wurden, die Probleme elastischer anisotropischer Platten untersucht.

Абстракт—В настоящей работе приьодится нелинейная теория однородных анизотропных упругих оболочек, разработанная без учёта теорий Лява-Кирхгоффа.

Граничные условия внешних поверхностей оболочки строго удовлетворены. Никакого огранияения не накладываетя на толщину оболочки, что позволяет ировести иссидование толстых люстинок и оболочек. В закиочении задача анизотропных уирущх пжстинок рассматриваетя ири потощи результатов, полученных в первой части, статби.